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Conditional Correctness of a Boundary Value Problem for a System of Parabolic Equations with Varying Direction of Time

Khajiev I.O.[1], Mamatkulova M.Sh.[2], Ro'zimatov J.A.[3]
[1]National University of Uzbekistan, Turin Polytechnic University in Tashkent, Universitet Str. 4, Tashkent, Uzbekistan
[2,3]National University of Uzbekistan, Universitet str. 4, Tashkent, Uzbekistan
E-mail: [1]kh.ikrom04@gmail.com, [2]mamatqulova_m@mail.ru, [3]rozimatovjakhongir@gmail.com

Abstract. In this work, the initial-boundary ill-posed problem for the system of parabolic equations with varying direction of time, which depend on each other, is studied. In this case, an a priori estimate of the solution of the problem is obtained, and the theorems of uniqueness and conditional stability are proved on the set of correctness of the solution.

Key words: ill-posed problem, a priori estimate, set of correctness, uniqueness theorem, conditional stability.

This paper is devoted to the study of the ill-posed initial-boundary value problem for a system of parabolic equations with varying direction of time.

In the domain $\Omega = \{-1 < x < 1,\ x \neq 0,\ 0 < t < T\}$ we consider the system of equations

$$\begin{cases} u_t + \operatorname{sgn} x\, u_{xx} + a_1 u + b_1 v = 0 \\ v_t + \operatorname{sgn} x\, v_{xx} + a_2 u + b_2 v = 0 \end{cases} \tag{1}$$

where a_1, a_2, b_1, b_2 - given real numbers, $(a_1 - a_2)^2 + 4b_1 b_2 > 0$, $b_i \neq 0,\ i = 1, 2$.

Statement of problem. Find a pair of functions $\big(u(x,t), v(x,t)\big)$ that satisfies the system of equations (1) and the following conditions:
the initial

$$u|_{t=0} = \varphi(x),\ v|_{t=0} = \psi(x), -1 \le x \le 1, \tag{2}$$

boundary

$$u|_{x=-1} = u|_{x=1} = 0, v|_{x=-1} = v|_{x=1} = 0,\ 0 \le t \le T \tag{3}$$

and gluing conditions

$$\begin{aligned} u|_{x=-0} &= u|_{x=+0},\ u_x|_{x=-0} = u_x|_{x=+0}, \\ v|_{x=-0} &= v|_{x=+0},\ v_x|_{x=-0} = v_x|_{x=+0}. \end{aligned} \tag{4}$$

Based on the theory of A.N. Tikhonov, the existence of a solution to this type of problem is assumed, and the main task is to prove the uniqueness and conditional stability of the solution.

In this paper, the problem (1)-(4) is studied for conditional stability in the set of correctness. In this case, an a priori estimate for the solution of the problem (1)-(4) is obtained by the method of logarithmic convexity, and the theorems of uniqueness and conditional stability in the set of correctness are proved.

We make the following substitution for the problem (1)-(4):

$$\begin{cases} u(x,t) = \dfrac{a_1+\lambda_2}{b_2\left(\lambda_1-\lambda_2\right)} e^{\lambda_1 t}\omega(x,t) + \dfrac{a_1+\lambda_1}{b_2\left(\lambda_1-\lambda_2\right)} e^{\lambda_2 t}\eta(x,t), \\ v(x,t) = \dfrac{1}{\left(\lambda_1-\lambda_2\right)} e^{\lambda_1 t}\omega(x,t) + \dfrac{1}{\left(\lambda_1-\lambda_2\right)} e^{\lambda_2 t}\eta(x,t), \end{cases} \tag{5}$$

where λ_1, λ_2 are the real roots of the quadratic equation $\lambda^2 + \left(a_1+a_2\right)\lambda + a_1a_2 - b_1b_2 = 0$. As a result, we come to the following problems depending on the functions $\omega(x,t)$ and $\eta(x,t)$, respectively.

Problem 1. Find a function $\omega(x,t)$ in the domain Ω, that satisfies the equation

$$\omega_t + \operatorname{sgn}(x)\omega_{xx} = 0 \tag{6}$$

and next the conditions

$$\omega\big|_{t=0} = \bar{\varphi}(x),\ -1 \le x \le 1, \tag{7}$$

$$\omega\big|_{x=-1} = \omega\big|_{x=1} = 0, \quad 0 \le t \le T, \tag{8}$$

$$\omega\big|_{x=-0} = \omega\big|_{x=+0},\ \omega_x\big|_{x=-0} = \omega_x\big|_{x=+0},\ 0 \le t \le T, \tag{9}$$

where $\bar{\varphi}(x) = (a_1+\lambda_1)\psi(x) - b_2\varphi(x)$.

Problem 2. Find a function $\eta(x,t)$ in the domain Ω, that satisfies the equation

$$\eta_t + \operatorname{sgn}(x)\eta_{xx} = 0$$

and next the conditions

$$\eta\big|_{t=0} = \bar{\psi}(x), -1 \le x \le 1,$$

$$\eta\big|_{x=-1} = \eta\big|_{x=1} = 0, \quad 0 \le t \le T,$$

$$\eta\big|_{x=-0} = \eta\big|_{x=+0},\ \eta_x\big|_{x=-0} = \eta_x\big|_{x=+0},\ 0 \le t \le T,$$

where $\bar{\psi}(x) = b_2\varphi(x) - (a_1+\lambda_2)\psi(x)$.

Suppose $\|u\|^2 = \left(u,u\right)$, where the dot product is $\left(u,v\right) = \int_{-1}^{1} u \cdot v\, dx$.

Lemma 1. We assume a pair of functions $\left(u(x,t), v(x,t)\right)$ satisfies the system of equations (1) and conditions (2)-(4) in the domain Ω, then

$$\|u\| \le 2\left(|A_1|\delta(t,\lambda_1) + |A_2|\delta(t,\lambda_2)\right),\ \|v\| \le 2|A_3|\left(\delta(t,\lambda_1) + \delta(t,\lambda_2)\right) \tag{10}$$

inequalities are valid, here

$$\delta(t,\lambda)=\left(|a_1+\lambda|\|\psi'(x)\|+|b_2|\|\varphi'(x)\|\right)^{1-\frac{t}{T}}\left(|a_1+\lambda|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\right)^{\frac{t}{T}}.$$

Proof. For the solution of the problem 1, we consider the function

$$f(t)=\int_{-1}^{1}\omega_x^2(x,t)dx.$$

We find the first and second derivatives of the function $f(t)$:

$$f'(t)=2\int_{-1}^{1}\omega_x\omega_{xt}dx=-2\int_{-1}^{1}\omega_{xx}\omega_t dx=2\int_{-1}^{1}\operatorname{sgn}(x)\omega_{xx}^2dx,$$

$$f''(t)=4\int_{-1}^{1}\operatorname{sgn}(x)\omega_{xx}\omega_{xxt}dx=-4\int_{-1}^{1}\omega_t\omega_{xxt}dx=4\int_{-1}^{1}\omega_{xt}^2dx.$$

Integration by parts, boundary conditions (8) and equation (6) were used to derive these equations.

Now we introduce the function $g(t)=\ln(f(t))$. Then, based on the Cauchy–Bunyakovsky inequality we get

$$g''(t)=\frac{f(t)f''(t)-(f'(t))}{f^2(t)}=\frac{4\int_{-1}^{1}\omega_x^2(x,t)dx\cdot\int_{-1}^{1}\omega_{xt}^2dx-\left(2\int_{-1}^{1}\omega_x\omega_{xt}dx\right)}{f^2(t)}\geq 0.$$

So, using properties of $g''(t)\geq 0$ and logarithmic convex function, we generate the estimate

$$\int_{-1}^{1}\omega_x^2(x,t)dx\leq\left(\int_{-1}^{1}\omega_x^2(x,0)dx\right)^{1-\frac{t}{T}}\left(\int_{-1}^{1}\omega_x^2(x,T)dx\right)^{\frac{t}{T}}$$

or

$$\|\omega_x(x,t)\|\leq\|\omega_x(x,0)\|^{1-\frac{t}{T}}\|\omega_x(x,T)\|^{\frac{t}{T}}. \tag{11}$$

(The proof of inequality (11) is based on the book [1]).

Similarly for the solution of the problem 2 we have

$$\|\eta_x(x,t)\|\leq\|\eta_x(x,0)\|^{1-\frac{t}{T}}\|\eta_x(x,T)\|^{\frac{t}{T}}. \tag{12}$$

We enter notations: $A_1=\dfrac{a_1+\lambda_2}{b_2(\lambda_1-\lambda_2)}$, $A_2=\dfrac{a_1+\lambda_1}{b_2(\lambda_1-\lambda_2)}$, $A_3=\dfrac{1}{(\lambda_1-\lambda_2)}$. Then, based on the equality (5), we will give confirmations

$$\|u_x(x,t)\|\leq|A_1|e^{\lambda_1 t}\|\omega_x(x,t)\|+|A_2|e^{\lambda_2 t}\|\eta_x(x,t)\|,$$

$$\|v_x(x,t)\|\leq|A_3|e^{\lambda_1 t}\|\omega_x(x,t)\|+|A_3|e^{\lambda_2 t}\|\eta_x(x,t)\|,$$

$$\omega=e^{-\lambda_1 t}\left((a_1+\lambda_1)v-b_2u\right),\ \ \vartheta=e^{-\lambda_2 t}\left(b_2u-(a_1+\lambda_2)v\right).$$

From these, from inequalities (11) and (12), we get estimates

$$\|u_x(x,t)\| \le |A_1|\left(|a_1+\lambda_1|\|\psi'(x)\|+|b_2|\|\varphi'(x)\|\right)^{1-\frac{t}{T}}\left(|a_1+\lambda_1|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\right)^{\frac{t}{T}}+$$
$$|A_2|\left(|a_1+\lambda_2|\|\psi'(x)\|+|b_2|\|\varphi'(x)\|\right)^{1-\frac{t}{T}}\left(|a_1+\lambda_2|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\right)^{\frac{t}{T}},$$

$$\|v_x(x,t)\| \le |A_3|\left(|a_1+\lambda_1|\|\psi'(x)\|+|b_2|\|\varphi'(x)\|\right)^{1-\frac{t}{T}}\left(|a_1+\lambda_1|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\right)^{\frac{t}{T}}+$$
$$|A_3|\left(|a_1+\lambda_2|\|\psi'(x)\|+|b_2|\|\varphi'(x)\|\right)^{1-\frac{t}{T}}\left(|a_1+\lambda_2|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\right)^{\frac{t}{T}}.$$

Based on the boundary conditions (3) and the fact that the inequalities $\|u(x,t)\| \le 2\|u_x(x,t)\|$, $\|v(x,t)\| \le 2\|v_x(x,t)\|$ are reasonable, the required inequality (10) arises.

We enter the set of correctness of the problem (1)-(4) in the form

$$M=\left\{\left(u(x,t),v(x,t)\right):\|u_x(x,T)\|+\|v_x(x,T)\|\le m,\ m<\infty\right\} \tag{13}$$

Theorem 1. Let the solution of the problem (1)-(4) exit and $(u,v)\in M$. Then it is unique.

Proof. Let the pairs of functions $(u_1(x,t),v_1(x,t))$ and $(u_2(x,t),v_2(x,t))$ be the solutions of the problem (1)-(4). We denote $u=u_1-u_2$, $v=v_1-v_2$. Then the pair of functions (u,v) satisfies the system of equations

$$\begin{cases} u_t+\operatorname{sgn} x\, u_{xx}+a_1u+b_1v=0, \\ v_t+\operatorname{sgn} x\, v_{xx}+a_2u+b_2v=0 \end{cases}$$

with conditions

$$u|_{t=0}=0,\ v|_{t=0}=0,\ -1\le x\le 1,$$
$$u|_{t=-1}=u|_{t=1}=0,\ v|_{t=-1}=v|_{t=1}=0,\ 0\le t\le T,$$
$$\left.\begin{matrix} u|_{x=-0}=u|_{x=+0},\ u_x|_{x=-0}=u_x|_{x=+0}, \\ v|_{x=-0}=v|_{x=+0},\ v_x|_{x=-0}=v_x|_{x=+0}, \end{matrix}\right. \quad 0\le t\le T,$$

Based on the initial condition and the result (10) of Lemma 1, we have

$$\|u(x,t)\|\le 0,\ \|v(x,t)\|\le 0.$$

From that we have $u(x,t)\equiv 0,\ v(x,t)\equiv 0$ for $\forall(x,t)\in\Omega$. This shows that $u_1(x,t)\equiv u_2(x,t),\ v_1(x,t)\equiv v_2(x,t)$. So, the solution of the problem (1)-(4) is unique.

We assume that the solution of the problem (1)-(4) for the $\varphi(x),\psi(x)$ initial exact data is $(u(x,t),v(x,t))$, the solution of the problem (1)-(4) for the $\varphi_\varepsilon(x),\psi_\varepsilon(x)$ approximate data is $(u_\varepsilon(x,t),v_\varepsilon(x,t))$.

Global Scientific Review

Theorem 2. Let the solution of the problem (1)-(4) exist, $(u(x,t),v(x,t)),(u_\varepsilon(x,t),v_\varepsilon(x,t))\in M$ and $\|\varphi'(x)-\varphi'_\varepsilon(x)\|\le\varepsilon$, $\|\psi'(x)-\psi'_\varepsilon(x)\|\le\varepsilon$. Then the inequalities

$$\|u-u_\varepsilon\|\le 2|A_1|\delta_\varepsilon(t,\lambda_1,m)+2|A_2|\delta_\varepsilon(t,\lambda_2,m),$$
$$\|v-v_\varepsilon\|\le 2|A_3|\left(\delta_\varepsilon(t,\lambda_1,m)+\delta_\varepsilon(t,\lambda_2,m)\right)$$

are valid, here $\delta_\varepsilon(t,\lambda,m)=\left(\varepsilon\left(|a_1+\lambda|+|b_2|\right)\right)^{\frac{T-t}{T}}\cdot\left(2m\left(|b_2|+|a_1+\lambda_1|\right)\right)^{\frac{t}{T}}$.

Proof. We enter notations: $U=u-u_\varepsilon$, $V=v-v_\varepsilon$. Then (U,V) satisfies the system of equations

$$\begin{cases}U_t+\operatorname{sgn}xU_{xx}+a_1U+b_1V=0\\ V_t+\operatorname{sgn}xV_{xx}+a_2U+b_2V=0\end{cases}\tag{14}$$

with the initial

$$U|_{t=0}=\varphi-\varphi_\varepsilon,\ \ V|_{t=0}=\psi-\psi_\varepsilon\tag{15}$$

and (3), (4) conditions. For the problem (14), (15) we perform the following calculations based on inequality (10):

$$\left\|(a_1+\lambda_1)\left(\psi'(x)-\psi'_\varepsilon(x)\right)-b_2\left(\varphi'(x)-\varphi'_\varepsilon(x)\right)\right\|\le$$
$$\le|a_1+\lambda_1|\|\psi'(x)-\psi'_\varepsilon(x)\|+|b_2|\|\varphi'(x)-\varphi'_\varepsilon(x)\|\le\varepsilon\left(|b_2|+|a_1+\lambda_1|\right)$$

and

$$\left\|b_2\left(\varphi'(x)-\varphi'_\varepsilon(x)\right)-(a_1+\lambda_2)\left(\psi'(x)-\psi'_\varepsilon(x)\right)\right\|\le$$
$$\le|b_2|\left\|\left(\varphi'(x)-\varphi'_\varepsilon(x)\right)\right\|+|a_1+\lambda_2|\|\psi'(x)-\psi'_\varepsilon(x)\|\le\varepsilon\left(|b_2|+|a_1+\lambda_2|\right).$$

Now, taking into account the conditions $(u(x,t),v(x,t)),(u_\varepsilon(x,t),v_\varepsilon(x,t))\in M$, we have inequalities

$$|a_1+\lambda_1|\|V_x(x,T)\|+|b_2|\|U_x(x,T)\|\le$$
$$|a_1+\lambda_1|\|v_x(x,T)-v_{\varepsilon x}(x,T)\|+|b_2|\|u_x(x,T)-u_{\varepsilon x}(x,T)\|\le 2m\left(|b_2|+|b_1+\lambda_1|\right),$$
$$|a_1+\lambda_2|\|v_x(x,T)\|+|b_2|\|u_x(x,T)\|\le 2m\left(|b_2|+|b_1+\lambda_2|\right)$$

From the above inequalities and the result of Lemma 1, it follows that

$$\|U\|\le 2\left(|A_1|\delta_\varepsilon(t,\lambda_1,m)+|A_2|\delta_\varepsilon(t,\lambda_2,m)\right),\ \ \|V\|\le 2|A_3|\left(\delta_\varepsilon(t,\lambda_1,m)+\delta_\varepsilon(t,\lambda_2,m)\right),$$

where $\delta_\varepsilon(t,\lambda,m)=\left(\varepsilon\left(|a_1+\lambda|+|b_2|\right)\right)^{\frac{T-t}{T}}\cdot\left(2m\left(|b_2|+|a_1+\lambda_1|\right)\right)^{\frac{t}{T}}$. From here, replacing the designations $U=u-u_\varepsilon$, $V=v-v_\varepsilon$, we get the required inequalities.

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Global Scientific Review